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Emergent Universe in Einstein-Gauss-Bonnet Theory

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Abstract In this work, Emergent Universe scenario has been developed in Einstein-Gauss-Bonnet (EGB) theory. The universe is chosen as homogeneous and isotropic FRW model and the matter in the universe has two components—the first one is a perfect fluid with barotropic equation of state $p = \omega\rho$ (ω, a constant) and the other component is a real or phantom (or tachyonic) scalar field. Various possibilities for the existence of emergent scenario has been discussed and the results are compared with those in Einstein gravity.

Keywords Emergent universe · Einstein-Gauss-Bonnet theory

1 Introduction

Avoidance of big bang singularity is at present a challenging issue in cosmology. Attempts have been made in the perspective of quantum gravity (specially in loop formalism) as well as in classical general relativity. The idea of emergent universe is the result of the search for singularity-free inflationary model in classical general relativity. So an emergent universe model can be defined as a singularity free universe which is ever existing with an almost static nature in the infinite past ($t \to -\infty$) and then evolves into an inflationary stage. In fact, an extension of the original Lemaitre-Eddington model can be termed as emergent universe.

However, there was singularity-free solution in the literature since 1967—Harrison [1] described a closed model of the universe with radiation, which coincides with Einstein static model in infinite past. But actual search for Emergent model of the universe was started (about 40 years back) by Ellis and collaborators [2, 3]. They formulated a closed model of the universe filled with two non-interacting fluids—one is a minimally coupled scalar field having self-interaction potential and the other is a perfect fluid with equation of state $p = \omega\rho$. In fact, they studied only the asymptotic behaviour to characterize the emergent

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scenario without finding exact analytic solutions. Then Mukherjee et al. [4] solved semi-classical equations in the Starobinsky model for flat FRW space-time and examined the features of emergent scenario. Subsequently, Mukherjee and others [5] were able to obtain nonsingular (i.e. geodesically complete) inflationary solution with a part of the matter in exotic form. Recently, Debnath [6] has formulated an emergent model of the universe for exotic matter in the form of phantom or tachyonic field. Banerjee et al. [7, 8] have shown a model of emergent universe in brane scenario while Campo et al. [9] have studied a model of emergent universe for self-interacting Brans-Dicke theory.

In the present work, we look for an emergent universe scenario in higher dimensional modified Einstein gravity as the model is capable of solving some of the conceptual issues of the standard big bang model. In particular, incorporating exotic matter namely phantom or tachyonic field, the emergent universe model is explored in the Gauss-Bonnet theory for closed, open and flat models. The consideration of higher dimensional theory is a standard assumption in high energy physics particularly after the recent development in String Theory. The classical analogue of the effective string theory is the low energy effective action, containing squares and higher powers of curvature terms. Also, similar higher derivative gravitational terms appear in the renormalization of quantum field theory in the curved space background. But as a result, the field equations become fourth or higher order and ghost terms appear. This difficulty was resolved by Lovelock by choosing a particular combination of higher powered curvature terms. He showed that the field equations still remain second order and there are no ghost terms. Here we consider only the first two terms of the Lovelock gravity namely the Einstein-Hilbert (EH) and Gauss-Bonnet (GB) terms and hence known as Einstein-Gauss-Bonnet (EGB) theory. The Gauss Bonnet term is important both from physical and geometrical point of view. It is the next order leading term in the α-expansion of the heterotic superstring theory (α^{-1} is the string tension) and also it plays a fundamental role in Chern-Simons gravitational theories. Geometrically, in five dimension, EH with GB term in the action give the most general Lagrangian for the second order field equations while in 4D, the Lagrangian corresponding to EH action is the most general one and GB term is only topological one, not affecting the dynamics.

The action of the EGB theory in $(n+1)$-dimension can be written as

$$S = \frac{1}{2}\left[\int d^{n+1}x\sqrt{-g}(R+\alpha R_{GB})\right] + S_m$$

where

$$R_{GB} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$$

is the GB term; α, the GB coupling parameter has the dimension of *length*^{2} and S_m is the matter action. Now, variation of the above action w.r.t. the metric tensor gives the modified Einstein equations:

$$G_{ab} - \alpha H_{ab} = T_{ab},$$

where

$$H_{ab} = 4R_{ap}R^{p}{}_{b} + 4R^{qr}R_{aqbr} - 2RR_{ab} - 2R^{qrp}{}_{a}R_{bqrp} + \frac{1}{2}g_{ab}R_{GB}$$

is the Lovelock tensor.

2 Basic Equations for Emergent Scenario in Einstein-Gauss-Bonnet Theory

The field equations for homogeneous and isotropic model of the universe in Einstein-Gauss-Bonnet theory are:

$$H^2 + \frac{k}{a^2} + \alpha\left(H^2 + \frac{k}{a^2}\right)^2 = \frac{16\pi G}{n(n-1)}(\rho + \rho_\phi) \tag{1}$$

and

$$\left[1 + 2\alpha\left(H^2 + \frac{k}{a^2}\right)\right]\left(\dot{H} - \frac{k}{a^2}\right) = -\frac{8\pi G}{n-1}[p + \rho + p_\phi + \rho_\phi] \tag{2}$$

where ρ and p are the energy density and pressure of a perfect fluid having equation of state $p = \omega\rho$, (ω, a constant), ρ_ϕ and p_ϕ are the energy-density and pressure of a scalar field ϕ having expressions

$$\rho_\phi = \frac{\varepsilon}{2}\dot{\phi}^2 + V(\phi) \tag{3}$$

$$p_\phi = \frac{\varepsilon}{2}\dot{\phi}^2 - V(\phi) \tag{4}$$

Here $V(\phi)$ is the potential for the scalar field ϕ and $\varepsilon = \pm 1$ corresponds to normal or phantom scalar field. In the above field equations, $a(t)$ is the scale factor, k $(= 0, \pm 1)$ is the curvature scalar, α is the Gauss-Bonnet coupling parameter, G is the universal gravitational constant and $H = \frac{\dot{a}}{a}$ is the Hubble parameter and $D = n + 1$ is the dimension of the space-time.

If we assume the two components of the matter field as non-interacting, then the energy conservation equations are given by

$$\dot{\rho} + nH\rho(1 + \omega) = 0 \tag{5}$$

and

$$\dot{\rho}_\phi + nH(\rho_\phi + p_\phi) = 0 \tag{6}$$

For emergent scenario the appropriate form of the scale factor will be [6]

$$a(t) = a_0[\beta + e^{\gamma t}]^m \tag{7}$$

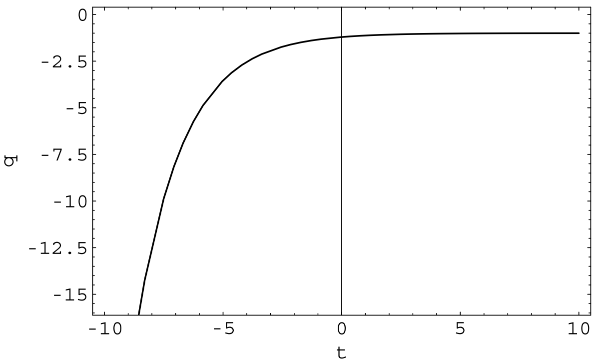
where a_0, β, γ, m are positive constants. The justification for such a choice are the following:

1. $a_0 > 0$ for scale factor to be positive definite.
2. $\beta > 0$, otherwise there will be big rip singularity.
3. $\gamma > 0$ and $m > 0$ for expanding model of the universe.
4. $\gamma < 0$ and $m < 0$ implies that there was a big bang singularity at infinite past.

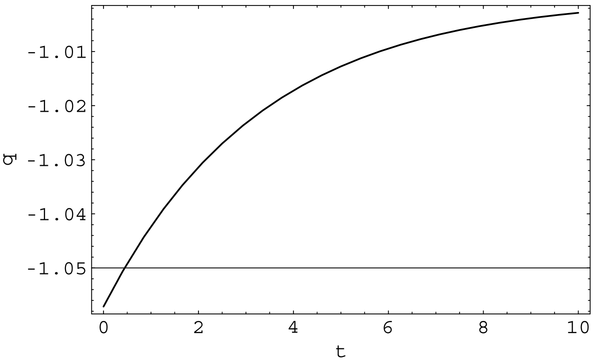
For the above choice of the scale factor, the universe started with a finite volume at $t = -\infty$, grows gradually without encountering any singularity for any t and finally, the universe will be of infinite volume at future infinity.

For this choice of the scale factor 'a', the Hubble parameter, it's derivatives and the deceleration parameter 'q' have the following expressions:

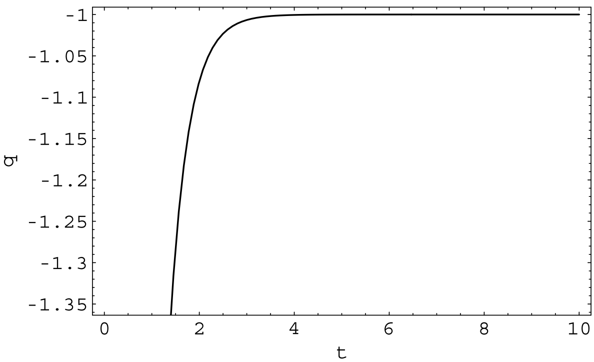
$$H = \frac{[m\gamma e^{\gamma t}]}{[\beta + e^{\gamma t}]},$$



(a)

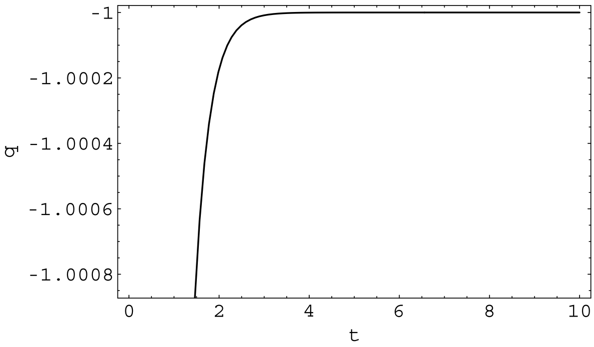


(b)



(c)

Fig. 1 (**a**) Represents the graph of q against t for the constants $\beta = 1.25$, $\gamma = 0.5$, $m = 6$; (**b**) represents the graph of q against t for the constants $\beta = 0.4$, $\gamma = 0.3$, $m = 7$; (**c**) represents the graph of q against t for the constants $\beta = 4$, $\gamma = 2.5$, $m = \frac{1}{3}$; (**d**) represents the graph of q against t for the constants $\beta = 7$, $\gamma = 3$, $m = 100$



(d)

Fig. 1 (*Continued*)

$$\dot{H} = \frac{[m\beta(\gamma^2)e^{\gamma t}]}{[\beta + e^{\gamma t}]^2}$$
$$\ddot{H} = \frac{[m\beta(\gamma^3)e^{\gamma t}(\beta - e^{\gamma t})]}{[\beta + e^{\gamma t}]^3}, \tag{8}$$
$$q = -1 - \frac{\beta}{[me^{\gamma t}]}$$

Note that H and $\dot{H}$ are positive definite while q is negative definite throughout the evolution. but $\ddot{H}$ changes sign at $t = \frac{\ln\beta}{\gamma}$. Asymptotically, as $t \to -\infty$, H, $\dot{H}$, and $\ddot{H}$ all tend to zero but $q \to -\infty$ while the model becomes a de Sitter universe as $t \to \infty$.

From the graphs of q versus t (Fig. 1a–1d) we see that q is an increasing function of t and as $t \to +\infty$, q asymptotically tends to -1.

Now using the equation of state for the perfect fluid the energy conservation relation (5) can be integrated to obtain

$$\rho = \rho_0 a^{-n(1+\omega)}, \tag{9}$$

where ρ_0 is an integration constant.

Also, writing the explicit form of ρ_ϕ and p_ϕ, field equations (1) and (2) take the form

$$H^2 + \frac{k}{a^2} + \alpha\left[H^2 + \frac{k}{a^2}\right]^2 = \frac{16\pi G}{n(n-1)}\left[\rho + \frac{\varepsilon}{2}\dot{\phi}^2 + V(\phi)\right] \tag{10}$$

and

$$\left[1 + 2\alpha\left(H^2 + \frac{k}{a^2}\right)\right]\left(\dot{H} - \frac{k}{a^2}\right) = \frac{-8\pi G}{n-1}\left[(1+\omega)\rho + \varepsilon\dot{\phi}^2\right] \tag{11}$$

Using (9) in (10) and (11) we have

$$(\dot{\phi})^2 = \frac{-(n-1)}{8\pi G\varepsilon}\left[\left\{1 + 2\alpha\left(H^2 + \frac{k}{a^2}\right)\right\}\left(\dot{H} - \frac{k}{a^2}\right)\right] - \frac{1+\omega}{\varepsilon}\rho_0 a^{-n(1+\omega)} \tag{12}$$

and

$$V(\phi)=\frac{n(n-1)}{16\pi G}\left[\left(H^2+\frac{k}{a^2}\right)\left\{1+\alpha\left(H^2+\frac{k}{a^2}\right)\right\}+\frac{1}{n}\left\{1+2\alpha\left(H^2+\frac{k}{a^2}\right)\right\}\right.$$
$$\left.\times\left(\dot{H}-\frac{k}{a^2}\right)\right]-\frac{(1-\omega)}{2}\rho_0 a^{-n(1+\omega)} \tag{13}$$

Note that $V(\phi)$ is independent of ε i.e. it has the same value for real or phantom scalar field.

3 Possibility of Emergent Scenario: The Restrictions

In this section, we discuss the possibility of emergent universe and present the restrictions for its validity. In other words, we examine whether the choice of $a(t)$ given by (7) is a possible solution for the field equations (10) and (11) for (i) the real scalar field and (ii) the phantom scalar field.

(i) Real scalar field ($\varepsilon=+1$)

In this case, the expression for $\dot{\phi}^2$ (i.e. (12)) becomes:

$$\dot{\phi}^2=\frac{-(n-1)}{8\pi G}\left[\left\{1+2\alpha\left(H^2+\frac{k}{a^2}\right)\right\}\left(\dot{H}-\frac{k}{a^2}\right)\right]-(1+\omega)\rho_0 a^{-n(1+\omega)}$$

We see that $k=0$ is not possible as the R.H.S. of the above equation becomes negative (i.e. $\dot{\phi}$ and hence ϕ becomes imaginary), so, the possible choices are $k=+1$ and $k=-1$. Thus emergent scenario is possible for real scalar field for closed model of the universe provided

$$\frac{1}{a^2}>\dot{H}+\frac{8\pi G(1+\omega)\rho_0 a^{-n(1+\omega)}}{(n-1)\{1+2\alpha(H^2+\frac{1}{a^2})\}} \tag{14}$$

Also, ϕ can be obtained as

$$\phi=\int\sqrt{\frac{(n-1)}{8\pi G}\left(\frac{1}{a^2}-\dot{H}\right)\left\{1+2\alpha\left(H^2+\frac{1}{a^2}\right)\right\}-(1+\omega)\rho_0 a^{-n(1+\omega)}}\,dt$$

As a is a given function of t, (see (7)), so in principle the above integral can be evaluated to obtain ϕ as a function of t.

Also, emergent scenario is possible for real scalar field for open model of the universe provided

$$\frac{1}{a^2}>H^2+\frac{1}{2\alpha}\left\{1+\frac{8\pi G(1+\omega)\rho_0 a^{-n(1+\omega)}}{(n-1)(\dot{H}+\frac{1}{a^2})}\right\} \tag{15}$$

Also, ϕ can be obtained as in the case of closed model and the resulting integral

$$\phi=\int\sqrt{\frac{(n-1)}{8\pi G}\left[\left(2\alpha\left(\frac{1}{a^2}-H^2\right)-1\right)\left(\dot{H}+\frac{1}{a^2}\right)\right]-(1+\omega)\rho_0 a^{-n(1+\omega)}}\,dt$$

can be evaluated to obtain ϕ as a function of t.

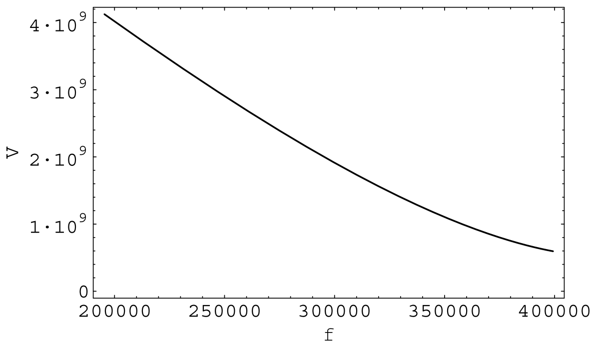


Fig. 2 The variation of V against ϕ (written as ‘f’ in figure) for normalizing the constants $\alpha = 9$, $\beta = 2$, $\gamma = 0.25$, $w = 1/2$, $n = 5$, $m = 1$, $\rho_0 = 4$, $a_0 = 0.5$ in real scalar field ($\varepsilon = 1, k = 1$)

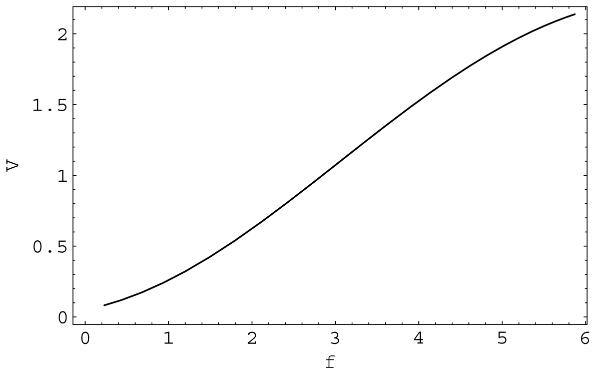


Fig. 3 The variation of V (in order of 10^{11}) against ϕ, written as ‘f’ in figure (in order of 10^5) for normalizing the constants $\alpha = 7$, $\beta = 2$, $\gamma = 0.5$, $w = 1/3$, $n = 5$, $m = 3$, $\rho_0 = 1$, $a_0 = 1$ in phantom scalar field

In Fig. 2 we have plotted V against ϕ for $\varepsilon = 1$ and $k = 1$ and the graph shows that V decreases as ϕ increases.

(ii) Phantom scalar field ($\varepsilon = -1$)

From (12), $\dot{\phi}^2$ can be written as

$$\dot{\phi}^2 = \frac{(n-1)}{8\pi G}\left[\left\{1 + 2\alpha(H^2 + \frac{k}{a^2})\right\}\left(\dot{H} - \frac{k}{a^2}\right)\right] + (1+\omega)\rho_0 a^{-n(1+\omega)}$$

As for the given choice of a (see (7)), $\dot{H}$ is always positive. So, the above equation is well defined for $k = 0$, and $k = -1$. For $k = +1$, to make the R.H.S. of the above equation positive definite we must have the restriction

$$\frac{1}{a^2} < \dot{H} + \frac{8\pi G(1+\omega)\rho_0 a^{-n(1+\omega)}}{(n-1)\{1 + 2\alpha(H^2 + \frac{1}{a^2})\}} \tag{16}$$

Similar to the real field case ϕ can be obtained as a function of t.

In Fig. 3, variation of V against ϕ for $\varepsilon = -1$ and $k = -1$ is shown. Here, V increases with ϕ.

4 Emergent Scenario for Tachyonic Scalar Field

For a tachyonic scalar field ψ, with potential $B(\psi)$, the energy density ρ_ψ and pressure p_ψ are given by

$$\rho_\psi = \frac{B(\psi)}{\sqrt{1-\varepsilon\dot{\psi}^2}} \tag{17}$$

and

$$p_\psi = -B(\psi)\sqrt{1-\varepsilon\dot{\psi}^2} \tag{18}$$

where as before $\varepsilon = \pm 1$ corresponds to normal and phantom tachyonic field. So, $\dot{\psi}^2$ and $B(\psi)$ can be written as

$$\dot{\psi}^2 = \frac{\rho_\psi + p_\psi}{\varepsilon(\rho_\psi)} \tag{19}$$

and

$$B(\psi) = \sqrt{-\rho_\psi p_\psi} \tag{20}$$

Thus for non-interacting perfect fluid and tachyonic scalar field the EGB field equations are

$$H^2 + \frac{k}{a^2} + \alpha\left(H^2 + \frac{k}{a^2}\right)^2 = \frac{16\pi G}{n(n-1)}[\rho + \rho_\psi] \tag{21}$$

and

$$\left\{1 + 2\alpha\left(H^2 + \frac{k}{a^2}\right)\right\}\left(\dot{H} - \frac{k}{a^2}\right) = \frac{-8\pi G}{(n-1)}[(1+\omega)\rho + \rho_\psi + p_\psi] \tag{22}$$

Hence $\dot{\psi}^2$ and $B(\psi)$ have the expressions

$$\dot{\psi}^2 = \left(\frac{-1}{\varepsilon}\right)\frac{\frac{n-1}{8\pi G}\{1 + 2\alpha(H^2 + \frac{k}{a^2})\}(\dot{H} - \frac{k}{a^2}) + (1+\omega)\rho}{\frac{(n(n-1))}{16\pi G}(H^2 + \frac{k}{a^2})\{1 + \alpha(H^2 + \frac{k}{a^2})\} - \rho} \tag{23}$$

and

$$B(\psi) = \sqrt{\frac{n(n-1)}{16\pi G}\left\{1 + \alpha\left(H^2 + \frac{k}{a^2}\right)\right\}\left(H^2 + \frac{k}{a^2}\right) - \rho} \\ \times\sqrt{\frac{n(n-1)}{16\pi G}\left(H^2 + \frac{k}{a^2}\right)\left\{1 + \alpha\left(H^2 + \frac{k}{a^2}\right)\right\} + \frac{n-1}{8\pi G}\left\{1 + 2\alpha\left(H^2 + \frac{k}{a^2}\right)\right\}\left(\dot{H} - \frac{k}{a^2}\right) + \omega\rho} \tag{24}$$

(i) Normal tachyonic scalar field ($\varepsilon = +1$)

In this case to make the R.H.S. of (23) to be positive definite we must have either the numerator or the denominator negative definite. But, the denominator cannot be negative because then the expression within the first square root in (24) becomes negative and hence

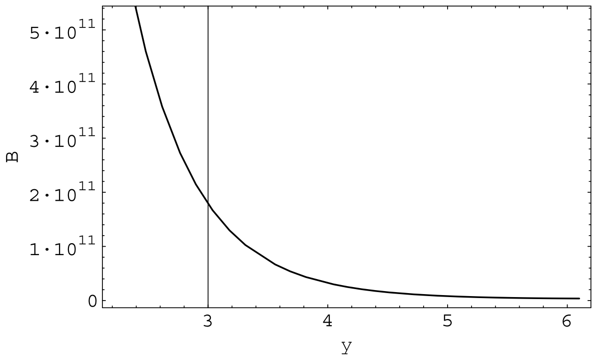


Fig. 4 The graph of B against ψ (written as 'y' in figure) for normalizing the constants $\alpha = 6$, $\beta = 0.5$, $\gamma = 0.5$, $w = 1/2$, $n = 5$, $m = 1$, $\rho_0 = 1$, $a_0 = 0.1$ in normal tachyonic field

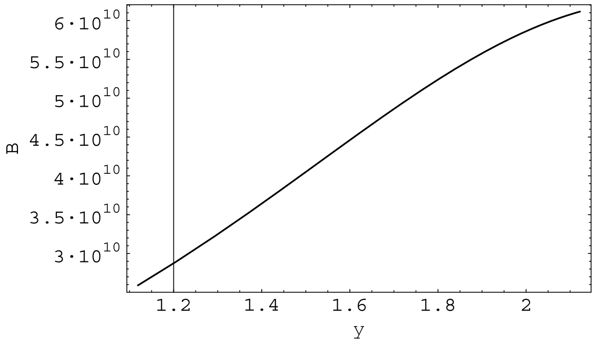


Fig. 5 The graph showing B versus ψ (written as 'y' in figure) for normalizing the constants $\alpha = 6$, $\beta = 1$, $\gamma = 0.5$, $w = 1/3$, $n = 6$, $m = 2$, $\rho_0 = 3$, $a_0 = 1$ in phantom tachyonic field $\varepsilon = -1$, $k = -1$

$B(\psi)$ becomes imaginary. So, the only possibility is that the numerator must be negative definite and this is possible for $k = +1$, provided:

$$\frac{1}{a^2} > \dot{H} + \frac{8\pi G(1+\omega)\rho_0 a^{-n(1+\omega)}}{(n-1)\{1+2\alpha(H^2+\frac{1}{a^2})\}} \tag{25}$$

Also to make the denominator positive definite, we must have

$$a^{-n(1+\omega)} < \frac{n(n-1)}{16\pi G\rho_0}\left(H^2+\frac{k}{a^2}\right)\left\{1+\alpha\left(H^2+\frac{k}{a^2}\right)\right\} \tag{26}$$

Also ψ can be obtained as:

$$\psi = \int \left[\frac{\frac{(n-1)}{8\pi G}\{1+2\alpha(H^2+\frac{1}{a^2})\}(\frac{1}{a^2}-\dot{H}) - (1+\omega)\rho_0 a^{-n(1+\omega)}}{\frac{n(n-1)}{16\pi G}(H^2+\frac{1}{a^2})\{1+\alpha(H^2+\frac{1}{a^2})\} - \rho_0 a^{-n(1+\omega)}}\right]^{\frac{1}{2}} dt$$

In Fig. 4, variation of B against ψ is shown for $\varepsilon = +1$, $k = +1$. B decreases with increasing ψ and becomes an asymptote to the ψ-axis.

(ii) Phantom tachyonic scalar field ($\varepsilon = -1$)

Here for real ψ both numerator and denominator in (23) must have the same sign. As stated earlier, the denominator cannot be negative, so numerator must be positive definite.

Note that for $k = 0, -1$, the numerator is always positive definite. However, for $k = +1$, in addition to restriction (26),we must have

$$\frac{1}{a^2} < \dot{H} + \frac{8\pi G(1+\omega)\rho_0 a^{-n(1+\omega)}}{(n-1)\{1+2\alpha(H^2+\frac{1}{a^2})\}} \tag{27}$$

In Fig. 5, we see that B increases as ψ increases.

5 Discussion

In this work, we have examined the possibility of emergent scenario for homogeneous and isotropic $(n+1)$ dimensional model of the universe filled with matter having non-interacting two components—one in the form of perfect fluid with linear equation of state $p = \omega\rho$ and a scalar field (real, phantom or tachyon) with potential as the other component. For real scalar field, the emergent universe model is possible when the space-time is closed and in addition we have restriction (14); and also when the space-time is open with restriction (15). However, for phantom scalar field, the emergent scenario is possible without any restriction for flat and open model while for closed model, it will be possible for restriction (16). For real tachyonic scalar field emergent scenario is possible for closed model when restrictions (25) and (26) hold true. For phantom tachyonic field emergent scenario is always possible for flat and open models and for closed model with restriction (27) in addition to (26). Further, one may note that the results do not depend sensitively on the Gauss-Bonnet Coupling parameter 'α'. Therefore, from the point of view of possibility of a emergent scenario, Einstein-Gauss-Bonnet gravity do not differ significantly from Einstein gravity. For future work, it will be attempted to consider the interaction between two components of the matter field.

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